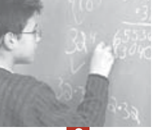
2

Main Features of the PISA Mathematics Theoretical Framework

This chapter provides a detailed description of the PISA 2003 assessment framework (OECD, 2003). It explains in detail the constructs of the mathematics assessment in PISA and lays out the context for the examples and further analysis presented in subsequent chapters.



INTRODUCTION

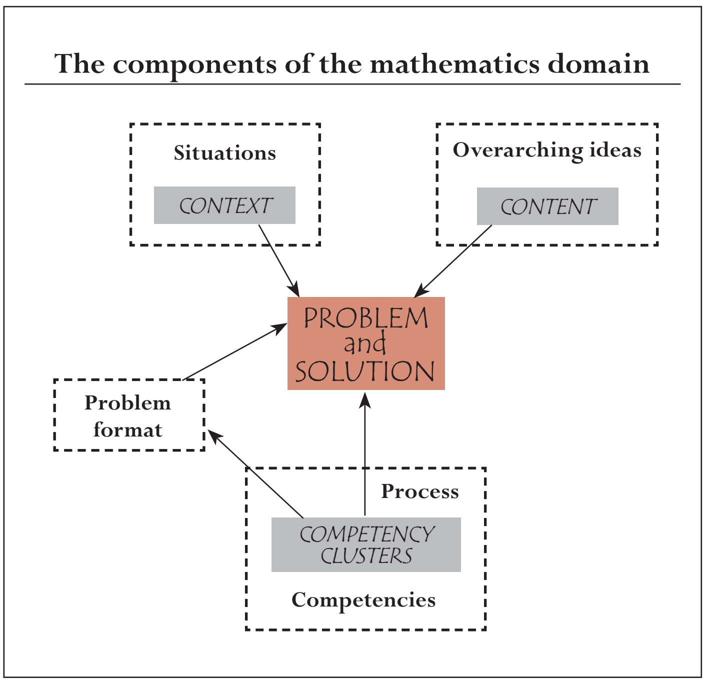
In order to appreciate and evaluate the mathematics items used in PISA it is important to understand the theoretical mathematics framework used for the assessment (OECD, 2003). This overview will focus on highlights of the framework, and illustrate these by means of PISA assessment items that have been released into the public domain.

Content, contexts, competencies and mathematical literacy *are the building blocks for the PISA* mathematics *framework.*

The structure of the PISA mathematics framework can be characterised by the mathematical representation: ML + 3Cs. ML stands for *mathematical literacy*, and the three Cs stand for content, contexts and competencies. Suppose a problem occurs in a situation in the real world; this situation provides a context for the mathematical task. In order to use mathematics to solve the problem, a student must have a degree of mastery over relevant mathematical content. And in order to solve the problem a solution process has to be developed and followed. To successfully execute these processes, a student needs certain competencies, which the framework discusses in three competency clusters.

This chapter begins with a discussion of *mathematical literacy*, and then outlines the three major components of the mathematics domain: context, content and competencies. These components can be illustrated schematically in Figure 2.1, reproduced from the framework (OECD, 2003).

Figure 2.1 ■ **Components of the PISA mathematics domain**



Source: OECD (2004a), Learning for Tomorrow's World: First Results from PISA 2003, OECD Publications, Paris.

MATHEMATICAL LITERACY

PISA defines a form of mathematical literacy…

The PISA *mathematical literacy* domain is concerned with the capacities of students to analyse, reason, and communicate ideas effectively as they pose, formulate, solve and interpret mathematical problems in a variety of situations. The accompanying assessment focuses on real-world problems, moving beyond the kinds of situations and problems typically encountered in school classrooms. In real-world settings, citizens regularly face situations when shopping, travelling, cooking, dealing with personal finances, analysing political positions, and considering other issues where the use of quantitative or spatial reasoning or other mathematical competencies would be of help in clarifying or solving a problem.

Such uses of mathematics are based on knowledge and skills learned and practised through the kinds of problems that typically appear in school textbooks and classrooms. However, these contextualised problems demand the ability to apply relevant skills in a less structured context, where the directions are not so clear for the students. Students have to make decisions about what knowledge may be relevant, what process or processes will lead to a possible solution, and how to reflect on the correctness and usefulness of the answer found.

… that requires engagement with mathematics…

Citizens in every country are increasingly confronted with a myriad of issues involving quantitative, spatial, probabilistic or relational reasoning. The media are full of information that use and misuse tables, charts, graphs and other visual representations to explain or clarify matters regarding weather, economics, medicine, sports, and environment, to name a few. Even closer to the daily life of every citizen are skills involving reading and interpreting bus or train schedules, understanding energy bills, arranging finances at the bank, economising resources, and making good business decisions, whether it is bartering or finding the best buy.

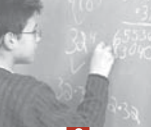
Thus, literacy in mathematics is about the functionality of the mathematics an individual learned at school. This functionality is an important survival skill for the citizen in today's information and knowledge society.

The definition of *mathematical literacy* for PISA is:

> Mathematical literacy *is an individual's capacity to identify, and understand, the role that mathematics plays in the world, to make well-founded judgments and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned, and reflective citizen*. (OECD 2003)

Some explanatory remarks are in order for this definition to become transparent.

In using the term "literacy", the PISA focus is on the sum total of mathematical knowledge a 15-year-old is capable of putting into functional use in a variety of contexts. The problems often call for reflective approaches involving insight



and some creativity. As such, PISA focuses on the mathematical knowledge and skills that go beyond the mathematics that has been defined within and limited to the outcomes of a school curriculum.

Mathematical literacy cannot be reduced to – but certainly presupposes – knowledge of mathematical terminology, facts and procedures as well as numerous skills in performing certain operations, and carrying out certain methods. PISA emphasises that the term "literacy" is not confined to indicating a basic, minimum level of functionality. On the contrary, PISA considers literacy as a continuous and multi-faceted spectrum ranging from aspects of basic functionality to high-level mastery.

… going beyond the mastery of mathematical techniques conventionally taught at school.

A crucial capacity implied by our notion of *mathematical literacy* is the ability to pose, formulate and solve intra- and extra-mathematical problems within a variety of domains and contexts. These range from purely mathematical ones to ones in which no mathematical structure is present from the outset but may be successfully introduced by the problem poser, problem solver, or both.

Attitudes and emotions (*e.g.* self-confidence, curiosity, feelings of interest and relevance, desire to do or understand things) are not components of the definition of *mathematical literacy*. Nevertheless they are important prerequisites for it. In principle it is possible to possess *mathematical literacy* without possessing such attitudes and emotions at the same time. In practice, however, it is not likely that such literacy will be exerted and put into practice by someone who does not have some degree of self-confidence, curiosity, feeling of interest and relevance, and desire to do or understand things that contain mathematical components.

The concept of *mathematical literacy* is by no means new. Related terms that have been used to describe it have varied from numeracy to quantitative literacy. Historically, Josiah Quincy connected the responsibility of citizens and lawmakers with statistical knowledge in 1816 and called it "political arithmetic". Since the identification of this linkage, attention has been given to the relation between the functionality of mathematics and needs of the responsible citizen. The definition of what constitutes *mathematical literacy* still varies widely from very narrow definitions like "the knowledge and skills required to apply arithmetic operations, either alone or sequentially, using numbers embedded in printed material" to "the ability to cope confidently with the mathematical needs of adult life" (Cohen, 2001).

Mathematical literacy is about dealing with "real" problems. That means that these problems are typically placed in some kind of a "situation". In short, the students have to "solve" a real world problem requiring them to use the skills and competencies they have acquired through schooling and life experiences. A fundamental process in this is referred to as "mathematisation". This process involves students shifting between the real-world context of the problem and the mathematical world needed to solve it. Mathematisation involves students in

interpreting and evaluating the problem and reflecting on its solution to ensure that the solution obtained indeed addresses the real situation that engendered the problem initially.

The process of "mathematisation" describes the ability of students to solve real-world problems by shifting between real-world and mathematical world contexts.

It is in this sense that *mathematical literacy* goes beyond curricular mathematics. Nevertheless, the assessment of *mathematical literacy* can't be separated from existing curricula and instruction because students' knowledge and skills largely depend on what and how they have learnt at school and on how this learning has been assessed. The analysis will continue through a discussion of the three Cs – content, context and competencies.

MATHEMATICAL CONTENT IN PISA – THE USE OF OVERARCHING IDEAS

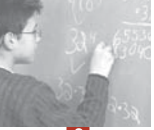
Mathematics school curricula are typically organised into topics and place an emphasis on procedures and formulas. This organisation sometimes makes it difficult for students to see or experience mathematics as a continuously growing scientific field that is constantly spreading into new fields and applications. Students are not positioned to see overarching concepts and relations, so mathematics appears to be a collection of fragmented pieces of factual knowledge.

"What is mathematics?" is not a simple question to answer. A person asked at random will most likely answer, "Mathematics is the study of numbers." Or, perhaps, "Mathematics is the science of numbers." And, as Devlin (1997) states in his book *Mathematics: The Science of Patterns*, the former is a huge misconception based on a description of mathematics that ceased to be accurate some 2 500 years ago. Present-day mathematics is a thriving, worldwide activity; it is an essential tool for many other domains like banking, engineering, manufacturing, medicine, social science, and physics. The explosion of mathematical activity that has taken place in the twentieth century has been dramatic.

At the turn of the nineteenth century, mathematics could reasonably be regarded as consisting of about a dozen distinct subjects: arithmetic, algebra, geometry, probability, calculus, topology, and so on. The typical present-day school curricula topics are drawn from this list.

A more reasonable figure for today, however, would be 70 to 80 distinct subjects. Some subjects (*e.g.* algebra, topology) have split into various sub fields; others (*e.g.* complexity theory, dynamical systems theory) are completely new areas of study (see, for example, the American Mathematical Society's *Mathematics by Classification*, 2009).

Mathematics can be seen as a language that describes patterns: patterns in nature and patterns invented by the human mind. Those patterns can either be real or imagined, static or dynamic, qualitative or quantitative, purely utilitarian or of little more than recreational interest. They can arise from the world



around us, from the depth of space and time, or from the inner workings of the human mind.

PISA organises mathematical content into four overarching ideas.

PISA aims to assess students' capacity to solve real problems, and therefore includes a range of mathematical content that is structured around different phenomena describing mathematical concepts, structures or ideas. This means describing mathematical content in relation to the phenomena and the kinds of problems for which it was created. In PISA these phenomena are called "overarching ideas". Using this approach PISA also covers a range of mathematical content that includes what is typically found in other mathematics assessments and in national mathematics curricula. However, PISA seeks to assess whether students can delve deeper to find the concepts that underlie all mathematics and therefore demonstrate a better understanding of the significance of these concepts in the world (for more information on phenomenological organisation of mathematical content see Steen, 1990).

The domain of mathematics is so rich and varied that it would not be possible to identify an exhaustive list of related phenomenological categories. PISA assesses four main overarching ideas:

- *Change and relationships*
- *Space and shape*
- *Quantity*
- *Uncertainty*

These four overarching ideas ensure the assessment of a sufficient variety and depth of mathematical content and demonstrate how phenomenological categories relate to more traditional strands of mathematical content.

Change and relationships

Change and relationships *involves the knowledge of mathematical manifestations of change, as well as functional relationships and dependency among variables.*

PISA recognises the importance of the understanding of *change and relationships* in *mathematical literacy*. Every natural phenomenon is a manifestation of change. Some examples are organisms changing as they grow, the cycle of seasons, the ebb and flow of tides, cycles for unemployment, weather changes, and the Dow-Jones index. Some of these change processes can be described or modelled by some rather straightforward mathematical functions (*e.g.* linear, exponential, periodic, logistic, either discrete or continuous). But many processes fall into different categories, and data analysis is quite often essential. The use of computer technology has resulted in more powerful approximation techniques, and more sophisticated visualisation of data. The patterns of change in nature and in mathematics do not in any sense follow the traditional mathematical content strands.

Following Stewart (1990), PISA is sensitive to the patterns of change and aims to assess how well students can:

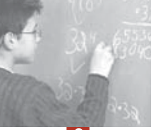
- represent changes in a comprehensible form;
- understand the fundamental types of change;
- recognise particular types of changes when they occur;
- apply these techniques to the outside world and
- control a changing universe to our best advantage.

The PISA overarching ideas of *change and relationships* includes many different traditional topics, most obviously functions and their representations, but also series. Further, *change and relationships*, as an overarching idea, encompasses patterns occurring in nature, art, and architecture in geometric situations.

Table 2.1 lists all of the released *change and relationships* questions that were used in the main PISA mathematics assessment and where readers can find these in Chapter 3. For example, the unit GROWING UP presents students with a graph showing the functional relationship between height in centimetres and age in years for a particular group of young males and young females. Question 1 invites students to interpret a statement about growth (change in height) over time, then to identify and carry out a simple calculation. Question 2 asks students to interpret the graph to identify the time period in which a certain relationship exists between heights of the females and males. Question 3 invites students to explain how the graph shows an aspect of change in growth rate.

Table 2.1
Examples of *change and relationships* questions

Question		***Where to find question in Chapter 3***
THE BEST CAR	*Question 1*	Examples of easy questions section
GROWING UP	*Question 1*	
GROWING UP	*Question 2*	
INTERNET RELAY CHAT	*Question 1*	Examples of questions of moderate difficulty section
GROWING UP	*Question 3*	
WALKING	*Question 1*	Examples of difficult questions section
INTERNET RELAY CHAT	*Question 2*	
THE BEST CAR	*Question 2*	
WALKING	*Question 3*	



Space and shape

PISA recognises that patterns are encountered not only in processes of *change and relationships*, but also can be explored in a static situation. Shapes are patterns: houses, churches, bridges, starfish, snowflakes, city plans, cloverleaves, crystals, and shadows. Geometric patterns can serve as relatively simple models of many kinds of phenomena, and their study is possible and desirable at all levels. Shape is a vital, growing, and fascinating theme in mathematics that has deep ties to traditional geometry (although relatively little in school geometry) but goes far beyond it in content, meaning, and method.

Space and shape *relates to the understanding of spatial and geometric phenomena and relationships.*

In the study of shape and constructions, students should look for similarities and differences as they analyse the components of form and recognise shapes in different representations and different dimensions. The study of shapes is closely knitted to "grasping space". That is learning to know, explore, and conquer in order to improve how we live, breathe, and move through the space in which we live (Freudenthal, 1973; Senechal, 1990).

Students must be able to understand relative positions of objects and to be aware of how they see things and why they see them this way. Students must learn to navigate through space and through constructions and shapes. Students should be able to understand the relation between shapes and images or visual representations (*e.g.* the relation between the real city and photographs or maps of the same city). They must also understand how three-dimensional objects can be represented in two dimensions, how shadows are formed and interpreted, and what "perspective" is and how it functions.

Described in this way, PISA recognises that the study of *space and shape* is open-ended, dynamic and fundamental to *mathematical literacy*. The *TWISTED BUILDING* unit is an example of a *space and shape* question that begins with the context of a geometric structure (a building), provides a more familiar mathematical representation of part of the situation, and calls on students to interpret the context, and to apply some mathematical knowledge to answer two questions examining spatial relationships from different perspectives (see Annex A1). Table 2.2 lists all of the released PISA *space and shape* questions that were used in the main PISA 2003 assessment and where the reader can find these in Chapter 3.

Table 2.2
Examples of *space and shape* questions

Question		*Where to find question in Chapter 3*
STAIRCASE	*Question 1*	Examples of easy questions section
CUBES	*Question 1*	
NUMBER CUBES	*Question 2*	Examples of questions of moderate difficulty section
CARPENTER	*Question 1*	Examples of difficult questions section

Quantity

PISA recognises the importance of quantitative literacy. In PISA, the overarching idea of *quantity* includes: meaning of operations, feel for magnitude of numbers, smart computations, mental arithmetic, estimations. Given the fundamental role of quantitative reasoning in applications of mathematics, as well as ubiquitous presence of numbers in our lives, it is not surprising that number concepts and skills form the core of school mathematics. In the earliest grade, mathematics teachers start children on a mathematical path designed to develop computational procedures of arithmetic together with the corresponding conceptual understanding that is required to solve quantitative problems and make informed decisions.

Quantity *requires an understanding of numeric phenomena, quantitative relationships and patterns.*

Quantitative literacy requires an ability to interpret numbers used to describe random as well as deterministic phenomena, to reason with complex sets of interrelated variables, and to devise and critically interpret methods for quantifying phenomena where no standard model exists.

Quantitatively literate students need a flexible ability to *(a)* identify critical relations in novel situations, *(b)* express those relations in effective symbolic form, *(c)* use computing tools to process information, and *(d)* interpret the results of these calculations (Fey, 1990).

PISA also aims to assess whether students can demonstrate creative quantitative reasoning. Creativity, coupled with conceptual understanding, is often ignored across the school curriculum. Students may have little experience in recognising identical problems presented in different formats or in identifying seemingly different problems that can be solved using the same mathematical tools. For example, in PISA quantitatively literate students would be able to recognise that the following three problems can all be solved using the concept of ratio:

- *Tonight you're giving a party. You want about a hundred cans of Coke. How many six-packs are you going to buy?*
- *A hang glider with glide ratio of 1 to 23 starts from a sheer cliff at a height of 123 meters. The pilot is aiming for a spot at a distance of 1 234 meters. Will she reach that spot?*
- *A school wants to rent minivans (with 8 seats each) to transport 78 students to a school camp. How many vans will the school need?*

Table 2.3 lists all of the released PISA *quantity* questions that were used in the main PISA 2003 assessment and where the reader can find these in Chapter 3. For example, the EXCHANGE RATE unit includes three questions with a context of travel and international exchange rates that call on students to demonstrate interpretation and quantitative reasoning skills.

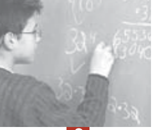


Table 2.3
Examples of *quantity* questions

Question		***Where to find question in Chapter 3***
EXCHANGE RATE	*Question 1*	Examples of easy questions section
EXCHANGE RATE	*Question 2*	
SKATEBOARD	*Question 1*	Examples of questions of moderate difficulty section
STEP PATTERN	*Question 1*	
BOOKSHELVES	*Question 1*	
SKATEBOARD	*Question 3*	
CHOICES	*Question 1*	
SKATEBOARD	*Question 2*	
EXCHANGE RATE	*Question 3*	

Uncertainty

Uncertainty *involves probabilistic and statistical phenomena as well as relationships that become increasingly relevant in the information society.*

In PISA the overarching idea of *uncertainty* is used to suggest two related topics: statistics and probability. Both of these are phenomena that are the subject of mathematical study. Recent moves have occurred in many countries towards increasing the coverage of statistics and probability within school curricula, particularly in recognition of the increasing importance of data in modern life. However it is particularly easy for a desire to increase the focus on data analysis to lead to a view of probability and statistics as a collection of specific and largely unrelated skills. Following the definition of the well-known statistics educator David S. Moore (1990), PISA *uncertainty* recognises the importance for students to: i) view data as *numbers in a context*; ii) develop an understanding of random events, the term he uses to label phenomena having uncertain individual outcomes but a regular pattern of outcomes in many repetitions.

Studies of human reasoning have shown that a student's intuition concerning randomness and chance profoundly contradicts the laws of probability. In part, this is due to students' limited contact with randomness. The study of data offers a natural setting for such an experience.

Randomness is a concept that is hard to deal with: children who begin their education with spelling and multiplication expect the world to be deterministic. They learn quickly to expect one answer to be right and others to be wrong, at least when the answers take numerical form. Probability is unexpected and uncomfortable, as Arthur Nielsen from the famous market research firm noted:

> [*Business people*] *accept numbers as representing Truth …. They do not see a number as a kind of shorthand for a range that describes our actual knowledge of the underlying condition. … I once decided that we would draw all charts to show a probable range around the number reported; for example, sales are either up three per cent, or down three per cent or somewhere in between. This turned out to be one of my dumber ideas. Our clients just couldn't work with this type of uncertainty* (Nielsen 1986, p. 8).

Statistical thinking involves reasoning from empirical data that are non-deterministic in nature, and should therefore be part of the mental equipment of every intelligent citizen. According to Moore (1990, p. 135) the core elements of statistical thinking involve the omnipresence of variation in processes and the need for data about processes to understand them. It also involves the need to take account of potential sources of variation when planning data collection or production, quantification of variation and explanation of variation.

Data analysis might help the learning of basic mathematics. The essence of data analysis is to "let the data speak" by looking for patterns in data, so that inferences can then be made about the underlying reality.

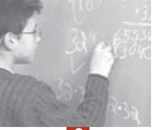
Table 2.4 lists all the released *uncertainty* questions that were used in the main PISA 2003 assessment and where the reader can find these in Chapter 3. For example, SUPPORT FOR THE PRESIDENT – QUESTION 1 exemplifies the statistics aspect of the *uncertainty* overarching idea. The stimulus for this question consists of information from opinion polls about a forthcoming election, conducted under varying conditions by four newspapers. Students were asked to reflect on the conditions under which the polls were conducted and to apply their understanding of such fundamental statistical concepts as randomness, and sampling procedures, and to tie these to their "common sense" ideas about polling procedures, to decide and explain which of the polls is likely to provide the best prediction.

Table 2.4
Examples of *uncertainty* questions

Question		*Where to find question in Chapter 3*
EXPORTS	*Question 1*	Examples of easy questions section
COLOURED CANDIES	*Question 1*	
LITTER	*Question 1*	
SCIENCE TESTS	*Question 1*	Examples of questions of moderate difficulty section
EARTHQUAKE	*Question 1*	
EXPORTS	*Question 2*	
SUPPORT FOR PRESIDENT	*Question 1*	
TEST SCORES	*Question 1*	Examples of difficult questions section
FORECAST OF RAIN	*Question 1*	
ROBBERIES	*Question 1*	

OVERARCHING IDEAS AND TRADITIONAL TOPICS

The comparison across PISA countries of student performance within each of the overarching ideas is described in the OECD report *Learning for Tomorrow's World: First Results from PISA 2003* (OECD, 2004a). This report also describes in great detail what students can typically do in these four areas of mathematics.



For certain topics and groups of countries, PISA mathematics questions are reclassified into five traditional curriculum topics: Number, Algebra, Measurement, Geometry and Data.

In Chapter 4 of the present report, the PISA items are reclassified to more traditional curriculum topics (used in the TIMSS survey). An analysis then is performed for specific mathematics topics and for specific groups of countries. In this study, the PISA mathematics items have been classified under the following five general mathematics curriculum topics: Number, Algebra, Measurement, Geometry and Data. These five curriculum topics are typically included in national curriculum documents in many countries. Table 2.5 shows the cross-classification of the 85 PISA 2003 main survey items according to the PISA and TIMSS (Grade 8) content classifications (OECD, 2003; Mullis, Martin, Gonzalez, and Chrostowski, 2004).

Table 2.5
Cross-tabulation of PISA items by PISA and traditional topics classifications

PISA Overarching ideas	Traditional topics					
	Algebra	Data	Geometry	Measurement	Number	Total
Change and relationships	7	10	0	2	3	22
Quantity	0	0	0	0	23	23
Space and shape	0	1	12	6	1	20
Uncertainty	0	15	0	0	5	20
Total	7	26	12	8	32	85

However it is important to view such a cross-classification with some caution, and to keep in mind that there is not a strict correspondence between the phenomenological categories used in the PISA framework to define mathematical content, and the traditional mathematics topics listed here and used in TIMSS. Such a list of topics (as well as others not listed here) has typically been used as a way to organise mathematical knowledge for the purposes of designing and delivering a school syllabus and for assessing the mastery of specific knowledge. The much broader PISA categories arise from the way mathematical phenomena appear in the real world – typically unaccompanied by any clues as to which pieces of mathematical knowledge might be relevant, and where a variety of different kinds of mathematical approaches might be possible and valid.

CONTEXT – SETTING THE MATHEMATICAL PROBLEM TO BE SOLVED

The context in which a mathematics problem is situated plays an important role in real world problem solving and *mathematical literacy*. The role and relevance of context is often underestimated and even ignored in school mathematics. PISA recognises the importance of context, and gives it a major role in the assessment of *mathematical literacy*.

Most importantly, PISA recognises the need to include a variety of contexts in the assessment, as well as allowing for a range of roles for the contexts. The variety is needed in such a large international assessment to minimise the chance of featuring issues and phenomena that are too culturally specific, or too unbalanced in relation to particular cultures.

Variety of contexts

PISA mathematics tasks are set in a range of contexts…

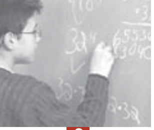
A wide range of contexts is encountered by citizens and it seems prudent to make use of the full range in constructing assessment tasks and in developing teaching and learning materials. The aspect of context needs further study, as results so far are inconclusive in the sense that one cannot say which contexts are more attractive for students or better suited for assessment or learning tasks. A common belief suggests that less able students "prefer" contexts closer to their immediate environment because they can engage with the context more readily. This can lead to items such as:

- *An ice cream seller has computed that for each 10 ice creams he sells, they will on average be the following kinds: 2 cups, 3 cones and 5 sticks. He orders 500 ice creams for the football game. How many of each kind should he order?*
- *Marge is lighter than Alice. Anny is lighter than Alice. Who is lighter: Anny or Marge?*
- *A pack of papers containing 500 sheets is 5 cm thick. How thick is one sheet of paper?*

… although there is debate as to whether or not contexts need to be close to the student.

At the primary level it is common to see this kind of context that is "close to the student" and taken from his or her "daily life". As students progress to the upper grades, the role of context often decreases and when it does occur, it is often as less familiar context drawn from the sciences or another discipline studied in the school curriculum. The exact role of context and its impact on student performance in assessment settings is not known. The use of items within context raises questions about differential student opportunities to learn the context interpretation skills. At the same time, the use of context situates the student assessment as close as possible to the real world contexts in which the student will be expected to make use of the mathematical content and modelling processes. Finding appropriate contexts and assuring they are bias free is a major issue related to context based assessment.

At the secondary level, an assumption that the context needs to be very close to the student does not necessarily hold. There are at least two relevant issues. First, it is necessary to recognise that there are more and more new "real-worlds" for students as their awareness and understanding grows – including the scientific and political worlds. But there also seems to be a tendency for this development to be postponed somewhat for lower-ability students. Second, it is necessary to recognise the role of context in assessments. Mathematics forms part of the real world, so students are bound to encounter mathematics to some degree and this is recognised in a small number of PISA items that have purely mathematical contexts (Linn, Baker, & Dunbar, 1991).



The four different contexts relate to…

PISA contexts

PISA mathematics questions are set in four different contexts:

- *Personal*
- *Educational and occupational*
- *Public*
- *Scientific* (including intra-mathematical)

… day-to-day activities…

The unit SKATEBOARD contains three questions, Q1, Q2 and Q3, classified in the *personal* context. The stimulus provides information about the cost of skateboard components, and the questions ask students to perform various calculations to explore costs and options related to constructing a skateboard from those components. It is assumed that such a context would be of immediate and direct personal relevance to many 15-year-olds.

… school and work situations…

The *educational and occupational* contexts include problem situations that students might confront while at school, including those rather artificial problems designed specifically for teaching or practice purposes, or problems that would be met in a work situation. STEP PATTERN Q1 is in the former category – it is a simple problem of number patterns that could typically be used to teach ideas about mathematical sequences. BOOKSHELVES Q1 could be regarded as an example of the latter category – the stimulus refers to the components needed by a carpenter to construct a set of bookshelves.

… the wider community…

Public contexts are those situations experienced in one's day-to-day interactions with the outside world. An example is ROBBERIES Q1 which presents an item from a newspaper, and asks students to make a judgment about claims made in the article.

… and scientific or explicitly mathematical problems …

Examples of items presented in a *scientific* context can be found in the unit titled DECREASING CO_2 LEVELS. This unit was used at the field trial stage but was not included in the main survey test instrument. The stimulus for this unit presents scientific data on the level of carbon dioxide emissions for several countries, and the items ask students to interpret and make use of the data presented.

Mathematical relevance of context

Contexts can be present just to make the problem look like a real-world problem (fake context, camouflage context, "zero-order" context). PISA attempts to stay away from such uses if possible in its assessment items, however some such problems have been used. An example is CARPENTER Q1. The context for the problem is a set of shapes being considered by a carpenter as possible borders around a hypothetical garden bed. Nothing about the carpenter or the garden are needed to understand or solve the problem, they merely provide the camouflage for a geometry problem.

PISA strives to present student with only relevant contexts…

The real "first-order" use of context is when the context is relevant and needed for solving the problem and judging the answer. An example of this use of context is in the questions Q1 and Q2 of the unit EXCHANGE RATE, where the context of two different currencies and the conversions between them are needed in order to understand and solve the problems and to evaluate the solutions. Another example is the problem of WATER TANK Q1. Here a water tank that comprises a conical part and a cylindrical part is presented, along with five optional graphs that could represent a mathematisation of the rate of change of water height in the tank over time as the tank is filled. Students must carefully check which of the given graphs fits the context.

… especially contexts that require students to shift between the mathematical problem and its context.

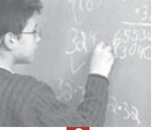
Second order use of context appears when one really needs to move backwards and forwards between the mathematical problem and its context in order to solve the problem or to reflect on the answer within the context to judge the correctness of the answer. This process is referred to as "mathematisation", and is discussed more extensively in Chapter 6. Thus, the distinction between first- and second-order uses of context lies in the role of the mathematisation process. In the first order, PISA has already pre-mathematised the problem, whereas in the second order much emphasis is placed on the mathematising process. An example of this second order use of context can be seen in TWISTED BUILDING Q1. The context of this question is a photograph of a computer model of a building, and students must impose their own mathematical structures on the situation in order to estimate the height of the building. Another example is ROCK CONCERT Q1 in which students are presented with the dimensions of a hypothetical football pitch, and have to model the space occupied by a person in order to estimate the number of football fans that could be accommodated.

The highest levels of *mathematical literacy* involve the ability to effectively handle such second-order contexts. This is the essence of real-world problem solving. From a mathematical instruction perspective, it is essential that students are exposed to activities that involve the purposeful interpretation of contexts in order to produce a relevant mathematical representation of the underlying problem, and that require reference to the context in order to produce a solution that addresses the problem.

Having dealt with content (mathematics literacy) and contexts (situations), the analysis turns to the third 'C': the competencies.

THE COMPETENCIES

An individual who is to make effective use of his or her mathematical knowledge within a variety of contexts needs to possess a number of mathematical competencies. Together, these competencies provide a comprehensive foundation for the proficiency scales that are described further in this chapter. To identify and examine these competencies, PISA has decided to make use of eight characteristic mathematical competencies that are relevant and meaningful across all education levels.



The cognitive activities needed to solve real-world problems are divided into eight characteristic mathematical competencies.

As can be seen immediately, there is substantial overlap among these competencies. This results from the ways in which these competencies are interrelated in the application of mathematics in solving a problem. It is usually necessary to draw on several of the competencies at once in such situations.

Mathematical thinking and reasoning

A fundamental mathematical competency is the capacity to think and reason mathematically. This involves asking probing and exploratory questions about what is possible, what could happen under certain conditions, how one might go about investigating a certain situation, and analysing logically the connections among problem elements.

Mathematical argumentation

Competency related to formal and logical argument and to justification and proof is also central to *mathematical literacy*. Such competence includes the ability to follow chains of reasoning and argument and to create such chains in analysing a process mathematically. At other times, this competence emerges in explaining, justifying or proving a result.

Modelling

Another competency associated with *mathematical literacy* is modelling. It is critical to *mathematical literacy* since it underpins the capacity to move comfortably between the real world in which problems are met and solutions are evaluated, and the mathematical world where problems are analysed and solved. The modelling process includes the capacity to structure the situation to be modelled, to translate from the real world to the mathematical world, to work with the model within the mathematical domain, to test and validate models used, to reflect critically on the model and its results especially in relation to the real world situation giving rise to the modelling activity, to communicate about the model, its results, and any limitations of such results, and to monitor and control the whole modelling process.

Problem posing and solving

An important step in solving problems is the capacity to define and clarify the problem to be solved. A mathematically literate person will have competence in working with problems in such a way that facilitates formulating clear problems from a relatively unstructured and ill-defined problem-situation and then carrying out sustained thought and analysis to bring relevant mathematical knowledge to bear on the transformed problem. This might involve recognising similarities with previously solved problems, or using insight to see where existing knowledge and skills can be applied, or creative linking of knowledge and information to produce a novel response to the situation.

Representation

A very basic competency that is critically important to *mathematical literacy* is the capacity to successfully use and manipulate a variety of different kinds of representations of mathematical objects and situations. This may include such representations as graphs, tables, charts, photographs, diagrams and text, as well as, algebraic and other symbolic mathematical representations. Central to this competence is the ability to understand and make use of interrelationships among these different representations.

Symbols and formalism

A defining competency of *mathematical literacy* is the capacity to understand and use mathematical symbolic language. This includes decoding symbolic language and understanding its connection to natural language. More generally, this competency also relates to the ability to handle and work with statements containing symbols and formulas, as well as the technical and procedural mathematical skills associated with a wide variety of formal mathematical processes.

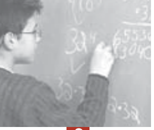
Communication

Mathematical literacy is also about competence in communication – understanding the written, oral, or graphical communications of others about mathematical matters and the ability to express one's own mathematical views in a variety of ways.

Aids and tools

The competency associated with knowing about and being able to make use of various aids and tools, including information technology tools, is another important part of *mathematical literacy*, particularly where mathematical instruction is concerned. Students need to recognise when different tools might be useful, to be able to make appropriate use of those tools, and to recognise the limitations of those tools.

It should be recognised that any effort to assess individual competencies is likely to result in artificial tasks and unnecessary and undesirable compartmentalisation of the *mathematical literacy* domain. In order to productively describe and report student's capabilities, as well as their strengths and weaknesses from an international perspective, some structure is needed.



COMPETENCY CLUSTERS

PISA competencies are classified into three clusters…

When doing real mathematics, it is necessary to draw simultaneously upon many of those competencies. In order to operationalise these mathematical competencies, PISA groups the underlying skills into three competency clusters:

1. Reproduction

2. Connections

3. Reflection

Reproduction cluster

… those involving familiar mathematical processes and computations …

Questions in the *reproduction* competency cluster require students to demonstrate that they can deal with knowledge of facts, recognise equivalents, recall mathematical objects and properties, perform routine procedures, apply standard algorithms and apply technical skills. Students also need to deal and operate with statements and expressions that contain symbols and formulas in "standard" form. Assessment items from the *reproduction* cluster are often in multiple-choice, fill-in-the-blank, matching, or (restricted) open-ended formats.

Table 2.6 lists all of the released questions in the *reproduction* competency cluster that were used in the PISA 2003 main assessment. Each of these is presented in full in Chapter 3 along with a description of the particular competencies that students need to draw upon to successfully solve these mathematical problems.

Table 2.6
Examples of questions in the *reproduction* competency cluster

Question		*Where to find question in Chapter 3*
EXCHANGE RATE	*Question 1*	Examples of easy questions section
STAIRCASE	*Question 1*	
EXPORTS	*Question 1*	
EXCHANGE RATE	*Question 2*	
THE BEST CAR	*Question 1*	
GROWING UP	*Question 1*	
GROWING UP	*Question 2*	
CUBES	*Question 1*	
SKATEBOARD	*Question 1*	Examples of questions of moderate difficulty section
STEP PATTERN	*Question 1*	
COLOURED CANDIES	*Question 1*	
SCIENCE TESTS	*Question 1*	
SKATEBOARD	*Question 2*	
WALKING	*Question 1*	Examples of difficult questions section

Connections cluster

Questions in the *connections* competency cluster require students to demonstrate that they can make linkages between the different strands and domains within mathematics and integrate information in order to solve simple problems in which students have choices of strategies or choices in their use of mathematical tools. Questions included in the *connections* competency cluster are non-routine, but they require relatively minor amounts of translation between the problem context and the mathematical world. In solving these problems students need to handle different forms of representation according to situation and purpose, and to be able to distinguish and relate different statements such as definitions, claims, examples, conditioned assertions and proof.

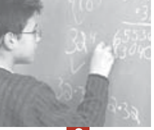
… those involving a degree of interpretation and linkages…

Students also need to show good understanding of mathematical language, including the decoding and interpreting of symbolic and formal language and understanding its relations to every-day language. Questions in the *connections* competency cluster are often placed within a *personal*, *public* or *educational and occupational* context and engage students in mathematical decision-making.

Table 2.7 lists the all of the released questions in the *connections* competency cluster that were used in the PISA 2003 main assessment. Each of these is presented in full in Chapter 3 along with a description of the particular competencies that students need to draw upon to successfully solve these mathematical problems.

Table 2.7
Examples of questions in the *connections* competency cluster

Question		*Where to find question in Chapter 3*
BOOKSHELVES	*Question 1*	Examples of questions of moderate difficulty section
NUMBER CUBES	*Question 2*	
INTERNET RELAY CHAT	*Question 1*	
SKATEBOARD	*Question 3*	
CHOICES	*Question 1*	
EXPORTS	*Question 2*	
GROWING UP	*Question 3*	
SUPPORT FOR PRESIDENT	*Question 1*	Examples of difficult questions section
TEST SCORES	*Question 1*	
FORECAST OF RAIN	*Question 1*	
ROBBERIES	*Question 1*	
WALKING	*Question 3*	
CARPENTER	*Question 1*	



Reflection cluster

Questions in the *reflection* competency cluster typically present students with a relatively unstructured situation, and ask them to recognise and extract the mathematics embedded in the situation and to identify and apply the mathematics needed to solve the problem. Students must analyse, interpret, develop their own models and strategies, and make mathematical arguments including proofs and generalisations. These competencies include a critical component that involves analysis of the model and reflection on the process.

... and those involving deeper insights and reflection.

Questions in the *reflection* competency cluster also require students to demonstrate that they can communicate effectively in different ways (*e.g.* giving explanations and arguments in written form, or perhaps using visualisations). Communication is meant to be a two-way process: students also need to be able to understand communications produced by others that have a mathematical component.

Table 2.8 lists all of the released questions in the *reflection* competency cluster that were used in the PISA 2003 main assessment. Each of these is presented in full in Chapter 3 along with a description of the particular competencies that students need to draw upon to successfully solve these mathematical problems.

Table 2.8
Examples of questions in the *reflection* competency cluster

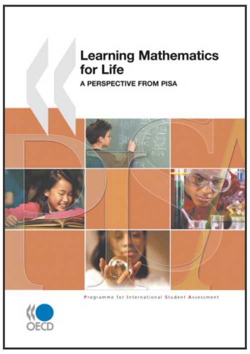
Question		*Where to find question in Chapter 3*
LITTER	*Question 1*	Examples of questions of moderate difficulty section
EARTHQUAKE	*Question 1*	
EXCHANGE RATE	*Question 3*	
INTERNET RELAY CHAT	*Question 2*	Examples of difficult questions section
THE BEST CAR	*Question 2*	

CONCLUSION

Mathematics in PISA 2003 focused on 15-year-olds' capabilities to use their mathematical knowledge in solving mathematical situations presented in a variety of settings. This focus was on their functional use of knowledge in solving real-life problems, rather than on ascertaining to what degree they had mastered their studies of formal mathematics or the degree to which they were facile with particular facts or procedures. This focus on *mathematical literacy* is examined through the lenses of student achievement related to situations calling for knowledge of *change and relationships*, *space and shape*, *quantity*, and *uncertainty*. These more global overarching ideas are supplemented by analyses taking the form of traditional topics such as algebra and functions, geometry and measurement, etc.

As the PISA 2003 study did not have teacher questionnaires providing information on the actual implementation of curriculum or teaching processes in the students' classrooms, PISA mathematics examines the student performance through the added lenses of context and competencies. Items were classified by context according to whether the situations contained were presented as dealing with their *personal* life, with a possible *educational* or *occupational* task, with a *public* use of mathematics task, or with an application of mathematics in a *scientific* or, even, mathematical setting. At the same time, items were classified by the demands they placed on students' cognitive processing capabilities. These demands were identified by the competencies discussed and their amalgamation into the clusters of *reproduction*, *connections*, and *reflection*.

The examination of the content through the filters of content, context, and competencies provides a filtering that helps understand the mathematical capabilities that students have developed in their first 15 years of life. Capabilities which are based in part on formal educational experiences, but which, in many cases, result from their direct experiences with solving problems that arise in daily life and decision making. This focus on *mathematical literacy* using the PISA definition strengthened by the three competencies presents a unique view of what students know and are able to do when confronted with situations to which mathematics knowledge and skills are applicable. The following chapters will detail the findings of the applications of this framework to the PISA 2003 mathematics results.



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